[6] Hall, B. D., and S. Spiegelman, these PROCEEDINGS, **47**, 137 (1961).
[7] Nygaard, A. P., and B. D. Hall, *Biochem. Biophys. Res. Comm.*, **12**, 98 (1963).
[8] Adams, M. H., *Bacteriophages* (New York: Interscience Publishers, Inc., 1959), p. 446.
[9] Kellenberger, G., M. L. Zichichi, and J. J. Weigle, these PROCEEDINGS, **47**, 869 (1961).
[10] Weigle, J., M. Meselson, and K. Paigen, *J. Mol. Biol.*, **1**, 379 (1960).
[11] Grossman, L., S. S. Levine, and W. S. Allison, *J. Mol. Biol.*, **3**, 47 (1961).
[12] Sagik, B. P., M. H. Green, M. Hayashi, and S. Spiegelman, *Biophys. J.*, 2, 409 (1962).
[13] Marmur, J., *J. Mol. Biol.*, **3**, 208 (1961).
[14] Chamberlin, M., and P. Berg, these PROCEEDINGS, **48**, 81 (1962).
[15] Weigle, J., personal communication.
[16] Yankofsky, S. A., and S. Spiegelman, these PROCEEDINGS, **48**, 1069 (1962).
[17] Burton, K., *Biochem. J.*, **62**, 315 (1956).
[18] Schildkraut, C. L., J. Marmur, and P. Doty, *J. Mol. Biol.*, **4**, 430 (1962).
[19] McCarthy, B. J., and E. T. Bolton, these PROCEEDINGS, **50**, 156 (1963).
[20] Luria, S. E., and J. W. Burrous, *J. Bact.*, **74**, 461 (1957).
[21] Kiger, J., and M. H. Green, in manuscript.
[22] Kellenberger, G., M. L. Zichichi, and J. Weigle, *Nature*, **187**, 161 (1960).
[23] Cairns, J., *J. Mol. Biol.*, **6**, 208 (1963), and personal communication.
[24] Burgi, E., and A. D. Hershey, *Biophys. J.*, **3**, 309 (1963).
[25] Geiduschek, E. P., T. Nakamoto, and S. B. Weiss, these PROCEEDINGS, **47**, 1405 (1961).
[26] Arber, W., *Arch. Sci.* (Geneva), **11**, 259 (1958).
[27] Lederberg, E. H., and J. Lederberg, *Genetics*, **38**, 51 (1953).
[28] McCarthy, B. J., and E. T. Bolton, these PROCEEDINGS, **50**, 156 (1963).
[29] Campbell, A., *Virology*, **11**, 339 (1960).
[30] Arber, W., and C. Lataste-Dorolle, *Path. Microbiol.*, **24**, 1012 (1961).
[31] Yoshikawa, H., and N. Sueoka, these PROCEEDINGS, **49**, 559 (1963).
[32] Nagata, T., these PROCEEDINGS, **49**, 551 (1963).
[33] Cowie, D. B., and B. J. McCarthy, these PROCEEDINGS, **50**, 537 (1963).
[34] Schildkraut, C. L., J. Marmur, and P. Doty, *J. Mol. Biol.*, **3**, 595 (1961).

ERGODIC THEORY, GROUP THEORY, AND DIFFERENTIAL GEOMETRY

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1. *Introduction.*—In this note we shall show how two programs announced earlier may be extended and partially unified by means of the concept of an "ergodic groupoid." The first of these programs[1] began with a generalization of the Laplace transform to locally compact commutative groups and involved a notion of analyticity for functions defined on the direct product of a connected locally compact commutative group with the vector space of all of its one-parameter subgroups. Some ten years later an extension of this first program was described[2] in which the dependence on group theory was removed, certain connections with ergodic theory were brought out, and differential notions other than analyticity were included. The second program grew out of our work on the problem of relating the unitary representations of a noncommutative separable locally compact group to those of a normal subgroup. The "ergodic case" of this problem suggests that every ergodic action of a separable locally compact group G be regarded as defining a "virtual

subgroup" of G which defines the action just as a closed subgroup H of G defines the transitive action of G on G/H. As described briefly on pages 652–654 of the notes to our 1961 colloquium lectures,[3] one can develop a theory of virtual subgroups which is parallel in many ways to the theory of locally compact groups.

The notion of ergodic groupoid (defined below) allows us to define a virtual group as a "similarity class" of ergodic groupoids and develop a theory of generalized groups which does not require them to occur as (virtual) subgroups of conventional groups. An ergodic groupoid for which every element is uniquely determined by its left and right units is said to be principal. Principal ergodic groupoids may be thought of as "ergodic equivalence relations." Refining and generalizing the notion of C^∞ ergodic lacing (defined in ref. 2), we are led to a notion which differs from that of C^∞ manifold in having "extra global structure." This extra global structure is defined by an ergodic equivalence relation, and the theory of virtual groups is applicable to it.

2. *Ergodic Groupoids and Equivalence Relations.*—Let S be a set. Let $\mathcal{E}$ be a subset of $S \times S$ which defines an equivalence relation in S. Let us introduce a multiplication in $\mathcal{E}$ by setting $(s_1,s_2)(s_3,s_4) = (s_1,s_4)$ whenever $s_2 = s_3$ and declaring the product to be undefined when $s_2 \neq s_3$. With this (not everywhere defined) multiplication, $\mathcal{E}$ satisfies all of the axioms for a Brandt groupoid[4] except the final one stating that any two units are the left and right units of a single element. It will be convenient to change Brandt's terminology (as has been done by others) and use the term "groupoid" to denote the more general object obtained by omitting the final axiom. (A groupoid may also be defined as an abstract category in which every map has an inverse.) The units of $\mathcal{E}$ are just the pairs s,s and so correspond one to-one to the elements of S. Moreover, $s_1,s_2 \in \mathcal{E}$ if and only if s_1,s_1 and s_2,s_2 are the left and right units, respectively, of the same element of $\mathcal{E}$. Thus, the pair $S,\mathcal{E}$ is determined by $\mathcal{E}$ and its groupoid structure. $\mathcal{E}$ is a *principal* groupoid in the sense that for a given ordered pair of units there is at most one groupoid element having these as left and right units, respectively. Clearly every principal groupoid is obtainable in this way from an equivalence relation. Now let $\mathfrak{F}$ be any groupoid which is also a (an analytic) Borel space.[5] We shall say that $\mathfrak{F}$ is a (an analytic) Borel groupoid if the two structures are so related that (i) the domain of definition $\mathfrak{D}$ of the product is a Borel subset of $\mathfrak{F} \times \mathfrak{F}$, (ii) $z_1, z_2 \rightarrow z_1z_2$ is a Borel function from $\mathfrak{D}$ to $\mathfrak{F}$, (iii) $z \rightarrow z^{-1}$ is a Borel function from $\mathfrak{F}$ to $\mathfrak{F}$. In any groupoid zz^{-1} is the left unit of z, and $z^{-1}z$ is the right unit of z. Thus, if $\pi(z) = zz^{-1}$, then π is a Borel function mapping $\mathfrak{F}$ onto the set $S_\mathfrak{F}$ of all units, and $S_\mathfrak{F}$ is a Borel subset of $\mathfrak{F}$. Let $\psi(z) = \pi(z),\pi(z^{-1})$. Then ψ is a Borel map of $\mathfrak{F}$ onto a subset of $S_\mathfrak{F} \times S_\mathfrak{F}$ which defines an equivalence relation in $S_\mathfrak{F}$. The mapping ψ is one-to-one if and only if $\mathfrak{F}$ is a principal groupoid and in any case is a homomorphism of $\mathfrak{F}$ onto the principal groupoid defined by the indicated equivalence relation. If G is a separable locally compact group and G acts on S so as to convert it into an analytic Borel G space, then $S \times G$ becomes an analytic Borel groupoid if $(s_1,x_1)(s_2,x_2)$ is defined when and only when $s_1x_1 = s_2$ and then is equal to (s_1,x_1x_2). This groupoid is principal if and only if $sx = s$ implies $x = e$ whenever $s \in S$ (e is the identity of G.)

Let μ be a finite measure[6] in the analytic Borel groupoid $\mathfrak{F}$. Let $\tilde{\mu}(E) = \mu(\pi^{-1}(E))$ for all Borel subsets E of $S_\mathfrak{F}$. As is well known,[7] there exists an assignment (unique almost everywhere with respect to $\tilde{\mu}$) of a measure μ_s to each $\pi^{-1}(s)$ such that for every Borel set $F \subset \mathfrak{F}$ we have $\mu(F) = \int_{S_{\mathfrak{F}}} \mu_s(F \cap \pi^{-1}(s))d\tilde{\mu}(s)$. Changing μ to another measure with the same null sets does not alter the null sets of $\tilde{\mu}$ or those of the μ_s. Thus, every measure class[8] C in $\mathfrak{F}$ defines a unique measure class $\tilde{C}$ in $S_{\mathfrak{F}}$ and (almost everywhere) a unique measure class C_s in $\pi^{-1}(s)$. Notice now that for each z in $\mathfrak{F}$, the mapping $x \rightarrow zx$ is a one-to-one Borel function from $\pi^{-1}(z^{-1}z)$ to $\pi^{-1}(zz^{-1})$. If there exists a null set N in $S_{\mathfrak{F}}$ such that for all z in $\mathfrak{F}$ with $zz^{-1} \notin N$ and $z^{-1}z \notin N$ this map carries the measure class $C_{z^{-1}z}$ onto the measure class $C_{zz^{-1}}$, we shall say that C is left invariant. If C is left invariant and invariant under $z \rightarrow z^{-1}$, we shall say that it is invariant. An invariant C will be said to be *ergodic* if no real valued Borel function f on S which is not $\tilde{C}$ almost everywhere constant can have the property that $f(z^{-1}z) = f(zz^{-1})$ for C almost all z in $\mathfrak{F}$. If C is invariant and ergodic, then the pair $\mathfrak{F}$, C will be called an *ergodic groupoid.* If the ergodic groupoid $\mathfrak{F},C$ is principal (so that $\mathfrak{F}$ is defined by an equivalence relation in $S_{\mathfrak{F}}$), we may obtain a measure class C_s' in the equivalence class of s by applying the mapping $z \rightarrow \pi(z^{-1})$ to the measure class C_s in $\pi^{-1}(s)$. Because C is invariant it follows that $C_{s_1}' = C_{s_2}'$ whenever s_1 and s_2 are equivalent. Thus, an invariant measure class C in $\mathfrak{F}$ defines a measure class $\tilde{C}$ in $S_{\mathfrak{F}}$ and a measure class in each equivalence class; conversely, $\tilde{C}$ and the C_s' determine C. When we wish to emphasize these facts about principal ergodic groupoids, we shall call them *ergodic equivalence relations.*

THEOREM 1. *Let G be a separable locally compact group and let S be a standard Borel G space. Make $S \times G$ into a Borel groupoid as indicated above. Let C_1 be a measure class in S and let C_2 be the measure class of Haar measure in G. Then $C_1 \times C_2$ is invariant with respect to the groupoid structure of $S \times G$ if and only if C_1 is invariant in S under the G action. If $C_1 \times C_2$ is invariant, then it is ergodic if and only if C_1 is ergodic in S under the G action.*

THEOREM 2. *Let $\mathfrak{F}$, C be an ergodic groupoid and let S_0 be a Borel subset of $S_{\mathfrak{F}}$ of positive $\tilde{C}$ measure. Let $\mathfrak{F} \upharpoonright S_0$ denote the set of all $z \in \mathfrak{F}$ with $\pi(z) \in S_0$ and $\pi(z^{-1}) \in S_0$. Then C restricted to $\mathfrak{F} \upharpoonright S_0$ converts it into an ergodic groupoid which is principal whenever $\mathfrak{F}$ is principal.*

Theorems 1 and 2 assure us of the existence of many examples of ergodic groupoids and equivalence relations. If the S_0 of Theorem 2 differs from S by a set of $\tilde{C}$ measure zero, we shall call $\mathfrak{F} \upharpoonright S_0$ an *inessential contraction* of $\mathfrak{F},C$.

3. *Virtual Groups.*—Let $\mathfrak{F}_1,C_1$ and $\mathfrak{F}_2,C_2$ be ergodic groupoids and let ϕ be a Borel function from $\mathfrak{F}_1$ to $\mathfrak{F}_2$. We shall say that ϕ is a *strict homomorphism* if (*i*) $\phi(z_1)\phi(z_2)$ is defined and equal to $\phi(z_1z_2)$ whenever z_1z_2 is defined, and (*ii*) if $S_{\mathfrak{F}_2}$ does not have a single equivalence class whose complement is of C_2 measure zero, then $\tilde{\phi}^{-1}(E)$ is a C_1 null set whenever E is a Borel subset of $S_{\mathfrak{F}_2}$ which is a C_2 null set. $\tilde{\phi}$ denotes the restriction of ϕ to $S_{\mathfrak{F}1}$. We shall say that ϕ is a *homomorphism* if its restriction to some inessential contraction is a strict homomorphism. Let ϕ be a (strict) homomorphism from $\mathfrak{F}_1,C_1$ to $\mathfrak{F}_2,C_2$ and let ψ be a (strict) homomorphism from $\mathfrak{F}_2,C_2$ to $\mathfrak{F}_3,C_3$. Then the composite mapping $\psi \circ \phi$ is easily seen to be a (strict) homomorphism of $\mathfrak{F}_1,C_1$ to $\mathfrak{F}_3,C_3$. We shall say that the strict homomorphisms ϕ_1 and ϕ_2 of $\mathfrak{F}_1,C_1$ into $\mathfrak{F}_2,C_2$ are strictly similar[9] and write $\varphi_1 \approx \varphi_2$ if there exists a Borel map θ of $S_{\mathfrak{F}1}$ into $\mathfrak{F}_2$ such that for all s in S the right and left units of $\theta(s)$ are $\phi_1(s)$ and $\phi_2(s)$, respectively, and $\theta(zz^{-1})\phi_1(z) = \phi_2(z)\theta(z^{-1}z)$ for all z in $\mathfrak{F}_1$. We shall say that two homomorphisms ϕ_1 and ϕ_2 are similar and write $\phi_1 \sim \phi_2$ if they have strictly similar restrictions to a common inessential contraction of $\mathfrak{F}_1,C_1$. It

is not hard to show that similarity and strict similarity are equivalence relations and that $\psi_1 \circ \phi_1$ is (strictly) similar to $\psi_2 \circ \phi_2$ if ϕ_1 and ψ_1 are (strictly) similar to ϕ_2 and ψ_2, respectively. Let ϕ_1 be a homomorphism of $\mathfrak{F}_1,C_1$ into $\mathfrak{F}_2,C_2$ and let ϕ_2 be a homomorphism of $\mathfrak{F}_2,C_2$ into $\mathfrak{F}_1,C_1$. If $\phi_2 \circ \phi_1$ and $\phi_1 \circ \phi_2$ are both similar to the relevant identities, we shall call the pair ϕ_1,ϕ_2 a *similarity* of $\mathfrak{F}_1,C_1$ with $\mathfrak{F}_.,C_2$ and say that the two ergodic groupoids are similar. By a *vii tual group* we shall mean a similarity class of ergodic groupoids. Similarities between ergodic groupoids "commute" with similarities between homomorphisms. Consequently, we may think of a similarity class of homomorphisms as being a homomorphism between the corresponding virtual groups. The following two theorems are actually special cases of a general theorem about homomorphisms of ergodic groupoids into groups. The general theorem will be formulated and proved in one of the detailed papers which will follow this announcement.

THEOREM 3. *Let H_1 and H_2 be closed subgroups of the separable locally compact groups G_1 and G_2. Let S_j denote the coset space G_j/H_j and convert $S_1 \times G_1$ and $S_2 \times G_2$ into ergodic groupoids as above. Then these ergodic groupoids are similar if and only if H_1 and H_2 are isomorphic as topological groups. Moreover, the similarity classes of homomorphisms of $S_1 \times G_1$ into $S_2 \times G_2$ correspond one-to-one in a natural way to the conjugacy clas<es of continuous homomorphisms of H_1 into H_2.*

THEOREM 4. *Let G be a separable locally compact group and let C_1 and C_2 be ergodic invariant measure classes in the analytic Borel G spaces S_1 and S_2. Let $\mathfrak{F}_j$ denote the groupoid associated with $S_j \times G$ so that $\mathfrak{F}_j,C_j$ is an ergodic groupoid for $j = 1,2$. Let ϕ_j denote the homomorphism $s,x \rightarrow x$ of $\mathfrak{F}_j,C_j$ into G. Suppose that there exists a similarity ψ_1,ψ_2 of $\mathfrak{F}_1,C_1$ with $\mathfrak{F}_2,C_2$ such that ϕ_1 is similar to $\phi_2 \circ \psi_1$. Then there exist invariant Borel null sets N_1 and N_2 in S_1 and S_2, respectively, and a one-to-one Borel mapping f of S_1-N_1 on S_2-N_2 such that $f(sx) = f(s)x$ for all $s,x \in S_1$-$N_1 \times G$.*

It follows from Theorem 4 that each ergodic action of G is determined to within an obvious equivalence by a certain virtual group and a homomorphism of this virtual group into G; thus, so to speak, by a "virtual" subgroup of G. Theorems 3 and 4 lend support to the view that virtual groups constitute a natural generalization of locally compact groups and may be expected to share many of their properties. The author plans to develop the theory of virtual groups in some detail in subsequent publications.

4. *Measures in Ergodic Equivalence Relations.*—Let $\mathcal{E}$ be a Borel subset of $S \times S$ defining an equivalence relation in the analytic Borel space S and let C be a measure class in $\mathcal{E}$ converting it into a principal ergodic groupoid. Let μ be a σ finite member of C and let ν be any σ-finite member of $\tilde{C}$. Applying the decomposition theorem referred to above to a finite member of C and working in an obvious manner with Radon Nikodym derivatives, one finds measures μ_s in the $\pi^{-1}(s)$ such that $\mu = \int \mu_s d\nu(s)$. The μ_s are determined uniquely almost everywhere by μ and ν; changing ν merely changes each μ_s by a multiplicative constant. We shall say that the pair μ,ν is *invariant* if $\mu_{s_1} = \mu_{s_2}$ for almost all pairs s_1,s_2 in $\mathcal{E}$, and μ is carried into itself by the map $s_1,s_2 \rightarrow s_2,s_1$.

THEOREM 5. *If μ,ν is an invariant pair, then μ and ν determine one another up to a multiplicative constant. If μ,ν is an invariant pair and σ is any positive Borel function, then μ',ν' is an invariant pair where $\nu'(E) = \int_E \sigma(s)d\nu(s)$ and $\mu'(F) = \int_F \sigma(s_1)\ \sigma(s_2)d\mu(s_1s_2)$.*

COROLLARY. *Let $s \rightarrow \mu_s^\circ$ be the assignment of measures to the equivalence classes of S defined by an invariant pair μ,ν. Then μ is determined by this assignment up to a multiplicative constant. Choosing this constant once and for all one has a "natural" one-to-one correspondence between the possible assignments $s \rightarrow \mu_s^\circ$ and the members ν of $\tilde{C}$.*

5. *A Generalization of C^∞ Manifolds.*—Let $S,\mathcal{E}$ be as in the preceding section and let each equivalence class in S under $\mathcal{E}$ be given the structure of a C^∞ manifold. Let $\mathcal{R}$ be the ring of all real valued Borel functions on S whose restriction to each equivalence class is a C^∞ function. Assume the given assignment to be such that $\mathcal{R}$ separates points so that the assignment is actually determined by $\mathcal{R}$. Let $C_s^{\mathcal{R}}$ denote the measure class in the equivalence class of s defined by its C^∞ manifold structure, that is, the class whose restriction to each sufficiently small neighborhood is that of the measures defined by the nonzero n forms (where n is the dimension of the manifold). Let C be a measure class in $\mathcal{E}$. We shall say that the quadruple $S,\mathcal{E},C,\mathcal{R}$ is a C^∞ *ergodic equivalence relation* if $S,\mathcal{E},C$ is an ergodic equivalence relation which admits an invariant pair and for $\tilde{C}$ almost all s in S we have $C_s = C_s^{\mathcal{R}}$.

One can construct a large class of examples as follows. Let $S,\mathcal{E},C$ be obtained from the action of a group G as indicated in Theorem 2. Let G be a Lie group and let $\tilde{C}$ contain a measure which is invariant under the group action. Then each equivalence class is a G orbit and inherits a C^∞ structure from that of G. It is not obvious that the ring $\mathcal{R}$ always separates points but it is easy to see that it does in numerous special cases, e.g., whenever S is a C^∞ manifold on which G acts smoothly.

Let $S,\mathcal{E},C,\mathcal{R}$ be a C^∞ ergodic equivalence relation. Then each $s \in S$ is contained in a unique C^∞ manifold M_s. Let V_s denote the tangent space to M_s at s. We shall call V_s the *tangent space to S at s.* We may now define vector and tensor fields on S just as in ordinary differential geometry. The contravariant vector field L taking s into L_s will be said to be a *Borel vector field* if for every $f \in \mathcal{R}$ the function $L(f)$ taking s into $L_s(f)$ is a Borel function on S. If $L(f)$ is a member of $\mathcal{R}$, we shall say that L is a C^∞ *vector field.* We define Borelness and being C^∞ for other kinds of vector and tensor fields in an analogous fashion. Many notions of ordinary differential geometry extend in an obvious fashion to C^∞ ergodic equivalence relations. For example, if f is a member of $\mathcal{R}$, then df is the C^∞ covariant tensor field (one form) such that $(df)_s(v) = f_v(s)$ where $v \in V_s$ and f_v is the action of v on f.

It is important to notice that a C^∞ ergodic equivalence relation is rather far from being the product of a measure space and a C^∞ manifold. This is partly because of the implications of ergodicity. For instance, let f be a member of $\mathcal{R}$ such that $df = 0$. Then f must be constant on each equivalence class and hence, by ergodicity, must be constant almost everywhere on S. To illustrate the assertion of the introduction about "extra global structure," let our C^∞ ergodic equivalence relation be such that each component C^∞ manifold is diffeomorphic to E^n. Let W be a C^∞-one form such that $dW = 0$. Then on each equivalence class there exists a C^∞ function f with $df =$ restriction of W to the class. The f for each class is unique up to an additive constant. To find a member F of $\mathcal{R}$ such that $dF = W$ we must choose these constants so that the resulting function on S is a Borel function on S. There are many examples showing that this is not always possible. Thus, the measure theoretic structure of $S,\mathcal{E},C$ can have the same kind of an effect on the global differential geometry of $S,\mathcal{E},C,\mathcal{R}$ as a nonvanishing first Betti number has on that of a C^∞ manifold.

In the integration of differential forms we have another respect in which C^∞ ergodic equivalence relations have global properties transcending those of the component C^∞ manifolds. Let ω be a Borel n form on S where n is the common dimension of the equivalence classes. Let it be positive with respect to an orientation. Then ω defines a unique member ν_s of each C_s, and ν_s depends only upon the equivalence class of s. Since $S,\mathcal{E},C$ admits an invariant pair, the assignment $s \rightarrow \nu_s$ is associated with an essentially unique member of $\tilde{C}$. Hence, modulo the choice of a single arbitrary constant, we obtain a natural one-to-one correspondence between the members of the measure class $\tilde{C}$ on S and the positive Borel n forms on S. Hence, it makes sense to speak of the integral of an n form over S or any Borel subset of S. More generally we can define C^∞ maps of m dimensional C^∞ ergodic equivalence relations into n dimensional ones and integrate m forms over such maps. It seems likely that one can formulate and prove an analogue of Stokes' theorem.

Given a Riemannian metric in a C^∞ ergodic equivalence relation, one has a natural associated positive n form and hence (up to a multiplicative constant) a natural measure μ in S. The formal Laplace Beltrami operator defined by the metric carries with it a corresponding notion of harmonic function and defines a symmetric operator in $\mathcal{L}^2(S,\mu)$. Interesting questions arise concerning the relationship between the spectral properties of the operator, the existence of harmonic functions on S, and the nature of the associated ergodic equivalence relation. In a similar manner practically every branch of analysis which is concerned with relating local differential properties to global ones suggests a family of questions about C^∞ ergodic equivalence relations. We hope to investigate some of these questions in later publications.

6. *Applications to Ergodic Theory.*—An ergodic flow has associated with it an ergodic equivalence relation and hence a virtual group. In many (if not in all) cases it will also be associated with a unique one-dimensional C^∞ ergodic equivalence relation. Group theory and differential geometry thus suggest a vast array of questions to ask about flows. One can hope that one will discover useful invariants of flows in this fashion and thus make progress in the difficult problem of classifying ergodic flows.

7. *Remarks.*—(*a*) In measure theoretic questions it is often useful to eliminate null sets by formulating everything in terms of the Boolean algebra obtained by factoring out the ideal of null sets from the Boolean algebra of all Borel subsets of a given space. On the other hand, in thus banishing points, one loses a conceptual and technical aid of some importance. Our strategy at the moment is to postpone a serious attempt to formulate everything in terms of Boolean algebras until the theory has been further developed and is better understood in its present form. Nevertheless, it is useful to keep the Boolean algebra point of view in mind, especially when one is doubtful about the "right way" to define a concept involving exceptional null sets.

(*b*) One can of course adapt the definitions of sections 2 and 3 to the groupoids defined by G spaces and equivalence relation spaces S where S is a topological space rather than a measure space. We have not yet explored the consequences of taking this point of view in topology in any detail. However, we have come upon one application which seems to be worth mentioning. Let $\mathfrak{U}$ be an open covering of a topological space X and let each member of the covering be such as to admit only trivial fiber bundles. Let X° denote the set of all pairs $\mathcal{O},x$ where $\mathcal{O} \in \mathfrak{U}, x \in \mathcal{O}$ and topologize X° so that for each fixed $\mathcal{O}$ the map $\mathcal{O},x \to x$ is a homeomorphism of its domain with $\mathcal{O}$ and the set of all $\mathcal{O},x$ is open in X°. Let us introduce an equivalence relation in X° by setting $\mathcal{O}_1,x_1$ equivalent to $\mathcal{O}_2,x_2$ whenever $x_1 = x_2$. Then X is homeomorphic to the space of all equivalence classes. Let $\mathcal{E}$ denote the set of ordered pairs defining this equivalence relation and regard $\mathcal{E}$ as a topological groupoid. Then the similarity classes of continuous homomorphisms of $\mathcal{E}$ into a topological group H correspond one-to-one in a natural way to the equivalence classes of principal bundles over X with group H. This fact is new at most in formulation. I am indebted to R. S. Palais for pointing out that it is simply a reformulation of the theorem connecting coordinate bundles and fiber bundles in section 2 of Steenrod's book.[10] On the other hand, it seems to be a useful way of looking at the connection in question and to be a suggestive way of thinking about fiber bundles and their properties. For example, if one decides to study the linear representation theory of the "virtual group" associated with the groupoid $\mathcal{E}$ and to construct the ring of all characters, one is led in a straightforward way to define the Grothendieck ring $K(X)$ of the space X. On the other hand, one must not press the group analogy too far in the case of similarity classes of topological groupoids. The ergodicity condition in section 3 plays a very important role, and similarity classes of groupoids are much less "grouplike" when it is missing.

(c) In his work on the foundations of differential geometry, C. Ehresmann has introduced a very general and abstract notion of local structure.[11] In defining and developing this notion he has been led to deal extensively with groupoids and to define topological groupoids, topological categories, and even groupoids and categories with additional structure defined by an abstract category. Though this work is different from ours in both aim and spirit (and moreover is not concerned with measure theory), there are points of contact. In particular, the device of turning a product $S \times G$ into a groupoid occurs in Ehresmann's work in the more general context in which G is a category "acting" on S and $S \times G$ is turned into another category. We are indebted to S. Sternberg for calling our attention to Ehresmann's work after reading a preliminary draft of this note. We are similarly indebted to R. J. Blattner for calling our attention to related work of Y. H. Clifton and J. W. Smith.[12] These authors are concerned with the following question. Let X be a manifold and let there be given a foliation of X. Let $\tilde{X}$ be the set of all leaves. Then the natural quotient topology in $\tilde{X}$ may be so degenerate that the only open sets are the empty set and the whole space. Can one find a substitute for $\tilde{X}$ that plays the role played by $\tilde{X}$ in the more tractable case in which the foliation is a fibering? Clifton and Smith propose a substitute which has a significant cohomology theory and reduces to $\tilde{X}$ when the foliation is a fibering. Now a C_∞ ergodic equivalence relation is a sort of foliation in which one has replaced the topological structure in the containing space by a measure theoretic one. The associated virtual group has space-like properties—in particular a cohomology theory. To the extent that this virtual group plays the role of the "topological object" of Clifton and Smith, there is a parallel between part of section 5 of this paper and the work of these authors. On the other hand, our virtual group is certainly different from their topological object—even in their context—since it does not depend on any topological structure in the underlying space. Moreover, it is possible that there are further differences. A closer study of the relationship would seem to be desirable. Apart from the parallel just described, the program announced in section 5 seems to have little in common with that apparently contemplated by Clifton and Smith.

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[1] Mackey, G. W., "The Laplace transform for locally compact abelian groups," these PROCEEDINGS, **34**, 156–162 (1948).

[2] Mackey, G. W., "The Laplace transform on groups and generalized analytic functions," Seminars on Analytic Functions, Air Force Office of Scientific Research, vol. 2 (1958), pp. 259–272.

[3] Published as "Infinite dimensional group representations," by G. W. Mackey, *Bull. Am. Math. Soc.*, **69**, 628–686 (1963).

[4] See Jacobson, N., *Theory of Rings* (New York: American Mathematical Society, 1943), pp. 132–133.

[5] See Mackey, G. W., "Borel structure in groups and their duals," *Trans. Amer. Math. Soc.*, **85**, 134–165 (1957), for definitions of this and related notions.

[6] In this note a measure is a countably additive set function taking values in the closed interval $0 \leq x \leq \infty$.

[7] See Theorem 11 of "Induced representations of locally compact groups, I," by G. W. Mackey, *Annals of Math.*, **55**, 101–139 (1952), for a formal statement with references to proofs.

[8] A measure class is the set of all σ finite measures having the same null sets as one of them.

[9] We have observed that a groupoid is a category with inverses. From this point of view a homomorphism is a functor, and (as pointed out to us by S. Eilenberg) strict similarity of homomorphisms is (apart from measure theoretic requirements) just natural equivalence of the corresponding functors.

[10] Steenrod, N. E., *The Topology of Fibre Bundles* (Princeton University Press, 1951).

[11] Ehresmann, C., "Gattungen von lokalen Strukturen," *Jber. Deutsch. Math. Verein*, **60**, 49–77 (1957), section 1.

[12] Clifton, Y. H., and J. W. Smith, "The category of topological objects," these PROCEEDINGS, **47**, 190–195 (1961); and "Topological objects and sheaves," *Trans. Amer. Math. Soc.*, 436–452 (1962).